

We saw in Chapter 6 that high-dimensional data has some peculiar characteristics, some of which are counterintuitive. For example, in high dimensions the center of the space is devoid of points, with most of the points being scattered along the surface of the space or in the corners. There is also an apparent proliferation of orthogonal axes. As a consequence high-dimensional data can cause problems for data mining and analysis, although in some cases high-dimensionality can help, for example, for nonlinear classification. Nevertheless, it is important to check whether the dimensionality can be reduced while preserving the essential properties of the full data matrix. This can aid data visualization as well as data mining. In this chapter we study methods that allow us to obtain optimal lower-dimensional projections of the data.

7.1 BACKGROUND

Let the data \mathbf{D} consist of n points over d attributes, that is, it is an $n \times d$ matrix, given as

$$\mathbf{D} = \begin{pmatrix} & X_1 & X_2 & \cdots & X_d \\ \mathbf{x}_1 & x_{11} & x_{12} & \cdots & x_{1d} \\ \mathbf{x}_2 & x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n & x_{n1} & x_{n2} & \cdots & x_{nd} \end{pmatrix}$$

Each point $\mathbf{x}_i = (x_{i1}, x_{i2}, \dots, x_{id})^T$ is a vector in the ambient d -dimensional vector space spanned by the d standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d$, where \mathbf{e}_i corresponds to the i th attribute X_i . Recall that the standard basis is an orthonormal basis for the data space, that is, the basis vectors are pairwise orthogonal, $\mathbf{e}_i^T \mathbf{e}_j = 0$, and have unit length $\|\mathbf{e}_i\| = 1$.

As such, given any other set of d orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$, with $\mathbf{u}_i^T \mathbf{u}_j = 0$ and $\|\mathbf{u}_i\| = 1$ (or $\mathbf{u}_i^T \mathbf{u}_i = 1$), we can re-express each point \mathbf{x} as the linear combination

$$\mathbf{x} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_d \mathbf{u}_d \quad (7.1)$$

where the vector $\mathbf{a} = (a_1, a_2, \dots, a_d)^T$ represents the coordinates of \mathbf{x} in the new basis. The above linear combination can also be expressed as a matrix multiplication:

$$\mathbf{x} = \mathbf{U}\mathbf{a} \quad (7.2)$$

where \mathbf{U} is the $d \times d$ matrix, whose i th column comprises the i th basis vector \mathbf{u}_i :

$$\mathbf{U} = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_d \\ | & | & \cdots & | \end{pmatrix}$$

The matrix \mathbf{U} is an *orthogonal* matrix, whose columns, the basis vectors, are *orthonormal*, that is, they are pairwise orthogonal and have unit length

$$\mathbf{u}_i^T \mathbf{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Because \mathbf{U} is orthogonal, this means that its inverse equals its transpose:

$$\mathbf{U}^{-1} = \mathbf{U}^T$$

which implies that $\mathbf{U}^T \mathbf{U} = \mathbf{I}$, where \mathbf{I} is the $d \times d$ identity matrix.

Multiplying Eq. (7.2) on both sides by \mathbf{U}^T yields the expression for computing the coordinates of \mathbf{x} in the new basis

$$\begin{aligned} \mathbf{U}^T \mathbf{x} &= \mathbf{U}^T \mathbf{U} \mathbf{a} \\ \mathbf{a} &= \mathbf{U}^T \mathbf{x} \end{aligned} \quad (7.3)$$

Example 7.1. Figure 7.1a shows the centered Iris dataset, with $n = 150$ points, in the $d = 3$ dimensional space comprising the sepal length (X_1), sepal width (X_2), and petal length (X_3) attributes. The space is spanned by the standard basis vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Figure 7.1b shows the same points in the space comprising the new basis vectors

$$\mathbf{u}_1 = \begin{pmatrix} -0.390 \\ 0.089 \\ -0.916 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -0.639 \\ -0.742 \\ 0.200 \end{pmatrix} \quad \mathbf{u}_3 = \begin{pmatrix} -0.663 \\ 0.664 \\ 0.346 \end{pmatrix}$$

For example, the new coordinates of the centered point $\mathbf{x} = (-0.343, -0.754, 0.241)^T$ can be computed as

$$\mathbf{a} = \mathbf{U}^T \mathbf{x} = \begin{pmatrix} -0.390 & 0.089 & -0.916 \\ -0.639 & -0.742 & 0.200 \\ -0.663 & 0.664 & 0.346 \end{pmatrix} \begin{pmatrix} -0.343 \\ -0.754 \\ 0.241 \end{pmatrix} = \begin{pmatrix} -0.154 \\ 0.828 \\ -0.190 \end{pmatrix}$$

One can verify that \mathbf{x} can be written as the linear combination

$$\mathbf{x} = -0.154\mathbf{u}_1 + 0.828\mathbf{u}_2 - 0.190\mathbf{u}_3$$

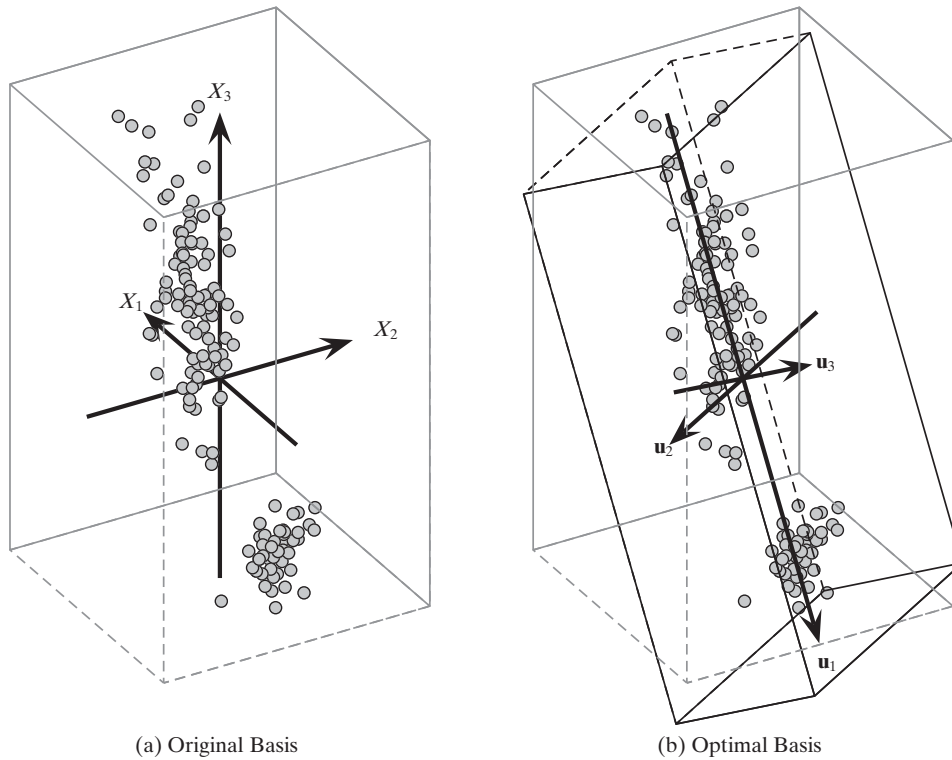


Figure 7.1. Iris data: optimal basis in three dimensions.

Because there are potentially infinite choices for the set of orthonormal basis vectors, one natural question is whether there exists an *optimal* basis, for a suitable notion of optimality. Further, it is often the case that the input dimensionality d is very large, which can cause various problems owing to the curse of dimensionality (see Chapter 6). It is natural to ask whether we can find a reduced dimensionality subspace that still preserves the essential characteristics of the data. That is, we are interested in finding the optimal r -dimensional representation of \mathbf{D} , with $r \ll d$. In other words, given a point \mathbf{x} , and assuming that the basis vectors have been sorted in decreasing order of importance, we can truncate its linear expansion [Eq. (7.1)] to just r terms, to obtain

$$\mathbf{x}' = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + \cdots + a_r \mathbf{u}_r = \sum_{i=1}^r a_i \mathbf{u}_i \quad (7.4)$$

Here \mathbf{x}' is the projection of \mathbf{x} onto the first r basis vectors, which can be written in matrix notation as follows:

$$\mathbf{x}' = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} = \mathbf{U}_r \mathbf{a}_r \quad (7.5)$$

where \mathbf{U}_r is the matrix comprising the first r basis vectors, and \mathbf{a}_r is vector comprising the first r coordinates. Further, because $\mathbf{a} = \mathbf{U}^T \mathbf{x}$ from Eq. (7.3), restricting it to the first r terms, we get

$$\mathbf{a}_r = \mathbf{U}_r^T \mathbf{x} \quad (7.6)$$

Plugging this into Eq. (7.5), the projection of \mathbf{x} onto the first r basis vectors can be compactly written as

$$\mathbf{x}' = \mathbf{U}_r \mathbf{U}_r^T \mathbf{x} = \mathbf{P}_r \mathbf{x} \quad (7.7)$$

where $\mathbf{P}_r = \mathbf{U}_r \mathbf{U}_r^T$ is the *orthogonal projection matrix* for the subspace spanned by the first r basis vectors. That is, \mathbf{P}_r is symmetric and $\mathbf{P}_r^2 = \mathbf{P}_r$. This is easy to verify because $\mathbf{P}_r^T = (\mathbf{U}_r \mathbf{U}_r^T)^T = \mathbf{U}_r \mathbf{U}_r^T = \mathbf{P}_r$, and $\mathbf{P}_r^2 = (\mathbf{U}_r \mathbf{U}_r^T)(\mathbf{U}_r \mathbf{U}_r^T) = \mathbf{U}_r \mathbf{U}_r^T = \mathbf{P}_r$, where we use the observation that $\mathbf{U}_r^T \mathbf{U}_r = \mathbf{I}_{r \times r}$, the $r \times r$ identity matrix. The projection matrix \mathbf{P}_r can also be written as the decomposition

$$\mathbf{P}_r = \mathbf{U}_r \mathbf{U}_r^T = \sum_{i=1}^r \mathbf{u}_i \mathbf{u}_i^T \quad (7.8)$$

From Eqs. (7.1) and (7.4), the projection of \mathbf{x} onto the remaining dimensions comprises the *error vector*

$$\boldsymbol{\epsilon} = \sum_{i=r+1}^d a_i \mathbf{u}_i = \mathbf{x} - \mathbf{x}'$$

It is worth noting that that \mathbf{x}' and $\boldsymbol{\epsilon}$ are orthogonal vectors:

$$\mathbf{x}'^T \boldsymbol{\epsilon} = \sum_{i=1}^r \sum_{j=r+1}^d a_i a_j \mathbf{u}_i^T \mathbf{u}_j = 0$$

This is a consequence of the basis being orthonormal. In fact, we can make an even stronger statement. The subspace spanned by the first r basis vectors

$$S_r = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_r)$$

and the subspace spanned by the remaining basis vectors

$$S_{d-r} = \text{span}(\mathbf{u}_{r+1}, \dots, \mathbf{u}_d)$$

are *orthogonal subspaces*, that is, all pairs of vectors $\mathbf{x} \in S_r$ and $\mathbf{y} \in S_{d-r}$ must be orthogonal. The subspace S_{d-r} is also called the *orthogonal complement* of S_r .

Example 7.2. Continuing Example 7.1, approximating the centered point $\mathbf{x} = (-0.343, -0.754, 0.241)^T$ by using only the first basis vector $\mathbf{u}_1 = (-0.390, 0.089, -0.916)^T$, we have

$$\mathbf{x}' = a_1 \mathbf{u}_1 = -0.154 \mathbf{u}_1 = \begin{pmatrix} 0.060 \\ -0.014 \\ 0.141 \end{pmatrix}$$

The projection of \mathbf{x} on \mathbf{u}_1 could have been obtained directly from the projection matrix

$$\begin{aligned}\mathbf{P}_1 &= \mathbf{u}_1 \mathbf{u}_1^T = \begin{pmatrix} -0.390 \\ 0.089 \\ -0.916 \end{pmatrix} \begin{pmatrix} -0.390 & 0.089 & -0.916 \end{pmatrix} \\ &= \begin{pmatrix} 0.152 & -0.035 & 0.357 \\ -0.035 & 0.008 & -0.082 \\ 0.357 & -0.082 & 0.839 \end{pmatrix}\end{aligned}$$

That is

$$\mathbf{x}' = \mathbf{P}_1 \mathbf{x} = \begin{pmatrix} 0.060 \\ -0.014 \\ 0.141 \end{pmatrix}$$

The error vector is given as

$$\boldsymbol{\epsilon} = a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 = \mathbf{x} - \mathbf{x}' = \begin{pmatrix} -0.40 \\ -0.74 \\ 0.10 \end{pmatrix}$$

One can verify that \mathbf{x}' and $\boldsymbol{\epsilon}$ are orthogonal, i.e.,

$$\mathbf{x}'^T \boldsymbol{\epsilon} = \begin{pmatrix} 0.060 & -0.014 & 0.141 \end{pmatrix} \begin{pmatrix} -0.40 \\ -0.74 \\ 0.10 \end{pmatrix} = 0$$

The goal of dimensionality reduction is to seek an r -dimensional basis that gives the best possible approximation \mathbf{x}'_i over all the points $\mathbf{x}_i \in \mathbf{D}$. Alternatively, we may seek to minimize the error $\boldsymbol{\epsilon}_i = \mathbf{x}_i - \mathbf{x}'_i$ over all the points.

7.2 PRINCIPAL COMPONENT ANALYSIS

Principal Component Analysis (PCA) is a technique that seeks a r -dimensional basis that best captures the variance in the data. The direction with the largest projected variance is called the first principal component. The orthogonal direction that captures the second largest projected variance is called the second principal component, and so on. As we shall see, the direction that maximizes the variance is also the one that minimizes the mean squared error.

7.2.1 Best Line Approximation

We will start with $r = 1$, that is, the one-dimensional subspace or line \mathbf{u} that best approximates \mathbf{D} in terms of the variance of the projected points. This will lead to the general PCA technique for the best $1 \leq r \leq d$ dimensional basis for \mathbf{D} .

Without loss of generality, we assume that \mathbf{u} has magnitude $\|\mathbf{u}\|^2 = \mathbf{u}^T \mathbf{u} = 1$; otherwise it is possible to keep on increasing the projected variance by simply

increasing the magnitude of \mathbf{u} . We also assume that the data has been centered so that it has mean $\boldsymbol{\mu} = \mathbf{0}$.

The projection of \mathbf{x}_i on the vector \mathbf{u} is given as

$$\mathbf{x}'_i = \left(\frac{\mathbf{u}^T \mathbf{x}_i}{\mathbf{u}^T \mathbf{u}} \right) \mathbf{u} = (\mathbf{u}^T \mathbf{x}_i) \mathbf{u} = a_i \mathbf{u}$$

where the scalar

$$a_i = \mathbf{u}^T \mathbf{x}_i$$

gives the coordinate of \mathbf{x}'_i along \mathbf{u} . Note that because the mean point is $\boldsymbol{\mu} = \mathbf{0}$, its coordinate along \mathbf{u} is $\mu_{\mathbf{u}} = 0$.

We have to choose the direction \mathbf{u} such that the variance of the projected points is maximized. The projected variance along \mathbf{u} is given as

$$\begin{aligned} \sigma_{\mathbf{u}}^2 &= \frac{1}{n} \sum_{i=1}^n (a_i - \mu_{\mathbf{u}})^2 \\ &= \frac{1}{n} \sum_{i=1}^n (\mathbf{u}^T \mathbf{x}_i)^2 \\ &= \frac{1}{n} \sum_{i=1}^n \mathbf{u}^T (\mathbf{x}_i \mathbf{x}_i^T) \mathbf{u} \\ &= \mathbf{u}^T \left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{u} \\ &= \mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} \end{aligned} \tag{7.9}$$

where $\boldsymbol{\Sigma}$ is the covariance matrix for the centered data \mathbf{D} .

To maximize the projected variance, we have to solve a constrained optimization problem, namely to maximize $\sigma_{\mathbf{u}}^2$ subject to the constraint that $\mathbf{u}^T \mathbf{u} = 1$. This can be solved by introducing a Lagrangian multiplier α for the constraint, to obtain the unconstrained maximization problem

$$\max_{\mathbf{u}} J(\mathbf{u}) = \mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} - \alpha (\mathbf{u}^T \mathbf{u} - 1) \tag{7.10}$$

Setting the derivative of $J(\mathbf{u})$ with respect to \mathbf{u} to the zero vector, we obtain

$$\begin{aligned} \frac{\partial}{\partial \mathbf{u}} J(\mathbf{u}) &= \mathbf{0} \\ \frac{\partial}{\partial \mathbf{u}} (\mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} - \alpha (\mathbf{u}^T \mathbf{u} - 1)) &= \mathbf{0} \\ 2\boldsymbol{\Sigma} \mathbf{u} - 2\alpha \mathbf{u} &= \mathbf{0} \\ \boldsymbol{\Sigma} \mathbf{u} &= \alpha \mathbf{u} \end{aligned} \tag{7.11}$$

This implies that α is an eigenvalue of the covariance matrix $\boldsymbol{\Sigma}$, with the associated eigenvector \mathbf{u} . Further, taking the dot product with \mathbf{u} on both sides of Eq. (7.11) yields

$$\mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} = \mathbf{u}^T \alpha \mathbf{u}$$

From Eq. (7.9), we then have

$$\begin{aligned}\sigma_{\mathbf{u}}^2 &= \alpha \mathbf{u}^T \mathbf{u} \\ \text{or } \sigma_{\mathbf{u}}^2 &= \alpha\end{aligned}\quad (7.12)$$

To maximize the projected variance $\sigma_{\mathbf{u}}^2$, we should thus choose the largest eigenvalue of Σ . In other words, the dominant eigenvector \mathbf{u}_1 specifies the direction of most variance, also called the *first principal component*, that is, $\mathbf{u} = \mathbf{u}_1$. Further, the largest eigenvalue λ_1 specifies the projected variance, that is, $\sigma_{\mathbf{u}}^2 = \alpha = \lambda_1$.

Minimum Squared Error Approach

We now show that the direction that maximizes the projected variance is also the one that minimizes the average squared error. As before, assume that the dataset \mathbf{D} has been centered by subtracting the mean from each point. For a point $\mathbf{x}_i \in \mathbf{D}$, let \mathbf{x}'_i denote its projection along the direction \mathbf{u} , and let $\boldsymbol{\epsilon}_i = \mathbf{x}_i - \mathbf{x}'_i$ denote the error vector. The mean squared error (MSE) optimization condition is defined as

$$\begin{aligned}MSE(\mathbf{u}) &= \frac{1}{n} \sum_{i=1}^n \|\boldsymbol{\epsilon}_i\|^2 \\ &= \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i - \mathbf{x}'_i\|^2 \\ &= \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i - \mathbf{x}'_i)^T (\mathbf{x}_i - \mathbf{x}'_i) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\|\mathbf{x}_i\|^2 - 2\mathbf{x}_i^T \mathbf{x}'_i + (\mathbf{x}'_i)^T \mathbf{x}'_i \right)\end{aligned}\quad (7.13)$$

Noting that $\mathbf{x}'_i = (\mathbf{u}^T \mathbf{x}_i) \mathbf{u}$, we have

$$\begin{aligned}&= \frac{1}{n} \sum_{i=1}^n \left(\|\mathbf{x}_i\|^2 - 2\mathbf{x}_i^T (\mathbf{u}^T \mathbf{x}_i) \mathbf{u} + ((\mathbf{u}^T \mathbf{x}_i) \mathbf{u})^T (\mathbf{u}^T \mathbf{x}_i) \mathbf{u} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\|\mathbf{x}_i\|^2 - 2(\mathbf{u}^T \mathbf{x}_i)(\mathbf{x}_i^T \mathbf{u}) + (\mathbf{u}^T \mathbf{x}_i)(\mathbf{x}_i^T \mathbf{u}) \mathbf{u}^T \mathbf{u} \right) \\ &= \frac{1}{n} \sum_{i=1}^n \left(\|\mathbf{x}_i\|^2 - (\mathbf{u}^T \mathbf{x}_i)(\mathbf{x}_i^T \mathbf{u}) \right) \\ &= \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i\|^2 - \frac{1}{n} \sum_{i=1}^n \mathbf{u}^T (\mathbf{x}_i \mathbf{x}_i^T) \mathbf{u} \\ &= \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i\|^2 - \mathbf{u}^T \left(\frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^T \right) \mathbf{u} \\ &= \sum_{i=1}^n \frac{\|\mathbf{x}_i\|^2}{n} - \mathbf{u}^T \Sigma \mathbf{u}\end{aligned}\quad (7.14)$$

(7.15)

Note that by Eq. (1.4) the total variance of the centered data (i.e., with $\boldsymbol{\mu} = \mathbf{0}$) is given as

$$\text{var}(\mathbf{D}) = \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i - \mathbf{0}\|^2 = \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i\|^2$$

Further, by Eq. (2.28), we have

$$\text{var}(\mathbf{D}) = \text{tr}(\boldsymbol{\Sigma}) = \sum_{i=1}^d \sigma_i^2$$

Thus, we may rewrite Eq. (7.15) as

$$MSE(\mathbf{u}) = \text{var}(\mathbf{D}) - \mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u} = \sum_{i=1}^d \sigma_i^2 - \mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u}$$

Because the first term, $\text{var}(\mathbf{D})$, is a constant for a given dataset \mathbf{D} , the vector \mathbf{u} that minimizes $MSE(\mathbf{u})$ is thus the same one that maximizes the second term, the projected variance $\mathbf{u}^T \boldsymbol{\Sigma} \mathbf{u}$. Because we know that \mathbf{u}_1 , the dominant eigenvector of $\boldsymbol{\Sigma}$, maximizes the projected variance, we have

$$MSE(\mathbf{u}_1) = \text{var}(\mathbf{D}) - \mathbf{u}_1^T \boldsymbol{\Sigma} \mathbf{u}_1 = \text{var}(\mathbf{D}) - \mathbf{u}_1^T \lambda_1 \mathbf{u}_1 = \text{var}(\mathbf{D}) - \lambda_1 \quad (7.16)$$

Thus, the principal component \mathbf{u}_1 , which is the direction that maximizes the projected variance, is also the direction that minimizes the mean squared error.

Example 7.3. Figure 7.2 shows the first principal component, that is, the best one-dimensional approximation, for the three dimensional Iris dataset shown in Figure 7.1a. The covariance matrix for this dataset is given as

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.681 & -0.039 & 1.265 \\ -0.039 & 0.187 & -0.320 \\ 1.265 & -0.320 & 3.092 \end{pmatrix}$$

The variance values σ_i^2 for each of the original dimensions are given along the main diagonal of $\boldsymbol{\Sigma}$. For example, $\sigma_1^2 = 0.681$, $\sigma_2^2 = 0.187$, and $\sigma_3^2 = 3.092$. The largest eigenvalue of $\boldsymbol{\Sigma}$ is $\lambda_1 = 3.662$, and the corresponding dominant eigenvector is $\mathbf{u}_1 = (-0.390, 0.089, -0.916)^T$. The unit vector \mathbf{u}_1 thus maximizes the projected variance, which is given as $J(\mathbf{u}_1) = \alpha = \lambda_1 = 3.662$. Figure 7.2 plots the principal component \mathbf{u}_1 . It also shows the error vectors $\boldsymbol{\epsilon}_i$, as thin gray line segments.

The total variance of the data is given as

$$\text{var}(\mathbf{D}) = \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i\|^2 = \frac{1}{150} \cdot 594.04 = 3.96$$

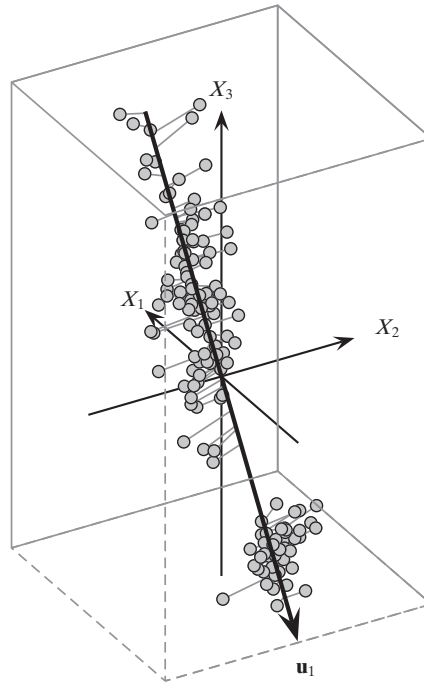


Figure 7.2. Best one-dimensional or line approximation.

We can also directly obtain the total variance as the trace of the covariance matrix:

$$\text{var}(\mathbf{D}) = \text{tr}(\mathbf{\Sigma}) = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0.681 + 0.187 + 3.092 = 3.96$$

Thus, using Eq. (7.16), the minimum value of the mean squared error is given as

$$\text{MSE}(\mathbf{u}_1) = \text{var}(\mathbf{D}) - \lambda_1 = 3.96 - 3.662 = 0.298$$

7.2.2 Best 2-dimensional Approximation

We are now interested in the best two-dimensional approximation to \mathbf{D} . As before, assume that \mathbf{D} has already been centered, so that $\boldsymbol{\mu} = \mathbf{0}$. We already computed the direction with the most variance, namely \mathbf{u}_1 , which is the eigenvector corresponding to the largest eigenvalue λ_1 of $\mathbf{\Sigma}$. We now want to find another direction \mathbf{v} , which also maximizes the projected variance, but is orthogonal to \mathbf{u}_1 . According to Eq. (7.9) the projected variance along \mathbf{v} is given as

$$\sigma_{\mathbf{v}}^2 = \mathbf{v}^T \mathbf{\Sigma} \mathbf{v}$$

We further require that \mathbf{v} be a unit vector orthogonal to \mathbf{u}_1 , that is,

$$\mathbf{v}^T \mathbf{u}_1 = 0$$

$$\mathbf{v}^T \mathbf{v} = 1$$

The optimization condition then becomes

$$\max_{\mathbf{v}} J(\mathbf{v}) = \mathbf{v}^T \boldsymbol{\Sigma} \mathbf{v} - \alpha (\mathbf{v}^T \mathbf{v} - 1) - \beta (\mathbf{v}^T \mathbf{u}_1 - 0) \quad (7.17)$$

Taking the derivative of $J(\mathbf{v})$ with respect to \mathbf{v} , and setting it to the zero vector, gives

$$2\boldsymbol{\Sigma} \mathbf{v} - 2\alpha \mathbf{v} - \beta \mathbf{u}_1 = \mathbf{0} \quad (7.18)$$

If we multiply on the left by \mathbf{u}_1^T we get

$$\begin{aligned} 2\mathbf{u}_1^T \boldsymbol{\Sigma} \mathbf{v} - 2\alpha \mathbf{u}_1^T \mathbf{v} - \beta \mathbf{u}_1^T \mathbf{u}_1 &= 0 \\ 2\mathbf{v}^T \boldsymbol{\Sigma} \mathbf{u}_1 - \beta &= 0, \text{ which implies that} \\ \beta &= 2\mathbf{v}^T \lambda_1 \mathbf{u}_1 = 2\lambda_1 \mathbf{v}^T \mathbf{u}_1 = 0 \end{aligned}$$

In the derivation above we used the fact that $\mathbf{u}_1^T \boldsymbol{\Sigma} \mathbf{v} = \mathbf{v}^T \boldsymbol{\Sigma} \mathbf{u}_1$, and that \mathbf{v} is orthogonal to \mathbf{u}_1 . Plugging $\beta = 0$ into Eq. (7.18) gives us

$$\begin{aligned} 2\boldsymbol{\Sigma} \mathbf{v} - 2\alpha \mathbf{v} &= \mathbf{0} \\ \boldsymbol{\Sigma} \mathbf{v} &= \alpha \mathbf{v} \end{aligned}$$

This means that \mathbf{v} is another eigenvector of $\boldsymbol{\Sigma}$. Also, as in Eq. (7.12), we have $\sigma_{\mathbf{v}}^2 = \alpha$. To maximize the variance along \mathbf{v} , we should choose $\alpha = \lambda_2$, the second largest eigenvalue of $\boldsymbol{\Sigma}$, with the *second principal component* being given by the corresponding eigenvector, that is, $\mathbf{v} = \mathbf{u}_2$.

Total Projected Variance

Let \mathbf{U}_2 be the matrix whose columns correspond to the two principal components, given as

$$\mathbf{U}_2 = \begin{pmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{pmatrix}$$

Given the point $\mathbf{x}_i \in \mathbf{D}$ its coordinates in the two-dimensional subspace spanned by \mathbf{u}_1 and \mathbf{u}_2 can be computed via Eq. (7.6), as follows:

$$\mathbf{a}_i = \mathbf{U}_2^T \mathbf{x}_i$$

Assume that each point $\mathbf{x}_i \in \mathbb{R}^d$ in \mathbf{D} has been projected to obtain its coordinates $\mathbf{a}_i \in \mathbb{R}^2$, yielding the new dataset \mathbf{A} . Further, because \mathbf{D} is assumed to be centered, with $\boldsymbol{\mu} = \mathbf{0}$, the coordinates of the projected mean are also zero because $\mathbf{U}_2^T \boldsymbol{\mu} = \mathbf{U}_2^T \mathbf{0} = \mathbf{0}$.

The total variance for \mathbf{A} is given as

$$\begin{aligned}
 \text{var}(\mathbf{A}) &= \frac{1}{n} \sum_{i=1}^n \|\mathbf{a}_i - \mathbf{0}\|^2 \\
 &= \frac{1}{n} \sum_{i=1}^n (\mathbf{U}_2^T \mathbf{x}_i)^T (\mathbf{U}_2^T \mathbf{x}_i) \\
 &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i^T (\mathbf{U}_2 \mathbf{U}_2^T) \mathbf{x}_i \\
 &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i^T \mathbf{P}_2 \mathbf{x}_i
 \end{aligned} \tag{7.19}$$

where \mathbf{P}_2 is the orthogonal projection matrix [Eq. (7.8)] given as

$$\mathbf{P}_2 = \mathbf{U}_2 \mathbf{U}_2^T = \mathbf{u}_1 \mathbf{u}_1^T + \mathbf{u}_2 \mathbf{u}_2^T$$

Substituting this into Eq. (7.19), the projected total variance is given as

$$\begin{aligned}
 \text{var}(\mathbf{A}) &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i^T \mathbf{P}_2 \mathbf{x}_i \\
 &= \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i^T (\mathbf{u}_1 \mathbf{u}_1^T + \mathbf{u}_2 \mathbf{u}_2^T) \mathbf{x}_i \\
 &= \frac{1}{n} \sum_{i=1}^n (\mathbf{u}_1^T \mathbf{x}_i) (\mathbf{x}_i^T \mathbf{u}_1) + \frac{1}{n} \sum_{i=1}^n (\mathbf{u}_2^T \mathbf{x}_i) (\mathbf{x}_i^T \mathbf{u}_2) \\
 &= \mathbf{u}_1^T \mathbf{\Sigma} \mathbf{u}_1 + \mathbf{u}_2^T \mathbf{\Sigma} \mathbf{u}_2
 \end{aligned} \tag{7.20}$$

Because \mathbf{u}_1 and \mathbf{u}_2 are eigenvectors of $\mathbf{\Sigma}$, we have $\mathbf{\Sigma} \mathbf{u}_1 = \lambda_1 \mathbf{u}_1$ and $\mathbf{\Sigma} \mathbf{u}_2 = \lambda_2 \mathbf{u}_2$, so that

$$\text{var}(\mathbf{A}) = \mathbf{u}_1^T \mathbf{\Sigma} \mathbf{u}_1 + \mathbf{u}_2^T \mathbf{\Sigma} \mathbf{u}_2 = \mathbf{u}_1^T \lambda_1 \mathbf{u}_1 + \mathbf{u}_2^T \lambda_2 \mathbf{u}_2 = \lambda_1 + \lambda_2 \tag{7.22}$$

Thus, the sum of the eigenvalues is the total variance of the projected points, and the first two principal components maximize this variance.

Mean Squared Error

We now show that the first two principal components also minimize the mean square error objective. The mean square error objective is given as

$$\begin{aligned}
 \text{MSE} &= \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i - \mathbf{x}'_i\|^2 \\
 &= \frac{1}{n} \sum_{i=1}^n \left(\|\mathbf{x}_i\|^2 - 2\mathbf{x}_i^T \mathbf{x}'_i + (\mathbf{x}'_i)^T \mathbf{x}'_i \right), \text{ using Eq. (7.14)} \\
 &= \text{var}(\mathbf{D}) + \frac{1}{n} \sum_{i=1}^n (-2\mathbf{x}_i^T \mathbf{P}_2 \mathbf{x}_i + (\mathbf{P}_2 \mathbf{x}_i)^T \mathbf{P}_2 \mathbf{x}_i), \text{ using Eq. (7.7) that } \mathbf{x}'_i = \mathbf{P}_2 \mathbf{x}_i
 \end{aligned}$$

$$\begin{aligned}
&= \text{var}(\mathbf{D}) - \frac{1}{n} \sum_{i=1}^n (\mathbf{x}_i^T \mathbf{P}_2 \mathbf{x}_i) \\
&= \text{var}(\mathbf{D}) - \text{var}(\mathbf{A}), \text{ using Eq. (7.20)}
\end{aligned} \tag{7.23}$$

Thus, the MSE objective is minimized precisely when the total projected variance $\text{var}(\mathbf{A})$ is maximized. From Eq. (7.22), we have

$$MSE = \text{var}(\mathbf{D}) - \lambda_1 - \lambda_2$$

Example 7.4. For the Iris dataset from Example 7.1, the two largest eigenvalues are $\lambda_1 = 3.662$, and $\lambda_2 = 0.239$, with the corresponding eigenvectors:

$$\mathbf{u}_1 = \begin{pmatrix} -0.390 \\ 0.089 \\ -0.916 \end{pmatrix} \quad \mathbf{u}_2 = \begin{pmatrix} -0.639 \\ -0.742 \\ 0.200 \end{pmatrix}$$

The projection matrix is given as

$$\begin{aligned}
\mathbf{P}_2 &= \mathbf{U}_2 \mathbf{U}_2^T = \begin{pmatrix} | & | \\ \mathbf{u}_1 & \mathbf{u}_2 \\ | & | \end{pmatrix} \begin{pmatrix} - & \mathbf{u}_1^T & - \\ - & \mathbf{u}_2^T & - \end{pmatrix} = \mathbf{u}_1 \mathbf{u}_1^T + \mathbf{u}_2 \mathbf{u}_2^T \\
&= \begin{pmatrix} 0.152 & -0.035 & 0.357 \\ -0.035 & 0.008 & -0.082 \\ 0.357 & -0.082 & 0.839 \end{pmatrix} + \begin{pmatrix} 0.408 & 0.474 & -0.128 \\ 0.474 & 0.551 & -0.148 \\ -0.128 & -0.148 & 0.04 \end{pmatrix} \\
&= \begin{pmatrix} 0.560 & 0.439 & 0.229 \\ 0.439 & 0.558 & -0.230 \\ 0.229 & -0.230 & 0.879 \end{pmatrix}
\end{aligned}$$

Thus, each point \mathbf{x}_i can be approximated by its projection onto the first two principal components $\mathbf{x}'_i = \mathbf{P}_2 \mathbf{x}_i$. Figure 7.3a plots this optimal 2-dimensional subspace spanned by \mathbf{u}_1 and \mathbf{u}_2 . The error vector $\boldsymbol{\epsilon}_i$ for each point is shown as a thin line segment. The gray points are behind the 2-dimensional subspace, whereas the white points are in front of it. The total variance captured by the subspace is given as

$$\lambda_1 + \lambda_2 = 3.662 + 0.239 = 3.901$$

The mean squared error is given as

$$MSE = \text{var}(\mathbf{D}) - \lambda_1 - \lambda_2 = 3.96 - 3.662 - 0.239 = 0.059$$

Figure 7.3b plots a nonoptimal 2-dimensional subspace. As one can see the optimal subspace maximizes the variance, and minimizes the squared error, whereas the nonoptimal subspace captures less variance, and has a high mean squared error value, which can be pictorially seen from the lengths of the error vectors (line segments). In fact, this is the worst possible 2-dimensional subspace; its MSE is 3.662.

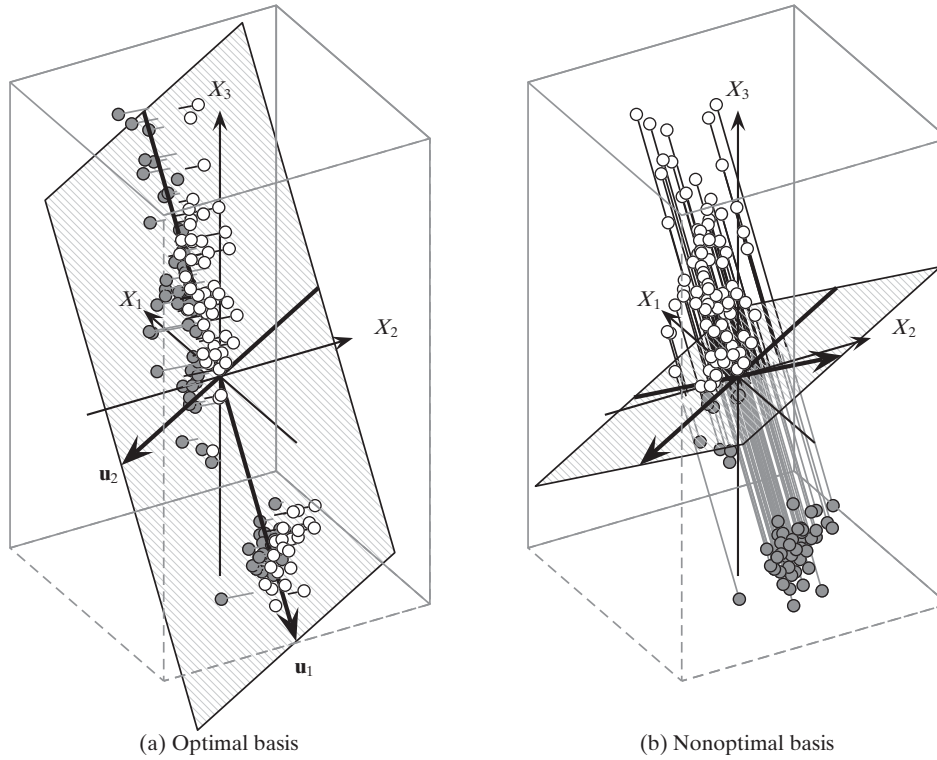


Figure 7.3. Best two-dimensional approximation.

7.2.3 Best r -dimensional Approximation

We are now interested in the best r -dimensional approximation to \mathbf{D} , where $2 < r \leq d$. Assume that we have already computed the first $j - 1$ principal components or eigenvectors, $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{j-1}$, corresponding to the $j - 1$ largest eigenvalues of Σ , for $1 \leq j \leq r$. To compute the j th new basis vector \mathbf{v} , we have to ensure that it is normalized to unit length, that is, $\mathbf{v}^T \mathbf{v} = 1$, and is orthogonal to all previous components \mathbf{u}_i , i.e., $\mathbf{u}_i^T \mathbf{v} = 0$, for $1 \leq i < j$. As before, the projected variance along \mathbf{v} is given as

$$\sigma_{\mathbf{v}}^2 = \mathbf{v}^T \Sigma \mathbf{v}$$

Combined with the constraints on \mathbf{v} , this leads to the following maximization problem with Lagrange multipliers:

$$\max_{\mathbf{v}} J(\mathbf{v}) = \mathbf{v}^T \Sigma \mathbf{v} - \alpha(\mathbf{v}^T \mathbf{v} - 1) - \sum_{i=1}^{j-1} \beta_i (\mathbf{u}_i^T \mathbf{v} - 0)$$

Taking the derivative of $J(\mathbf{v})$ with respect to \mathbf{v} and setting it to the zero vector gives

$$2\Sigma \mathbf{v} - 2\alpha \mathbf{v} - \sum_{i=1}^{j-1} \beta_i \mathbf{u}_i = \mathbf{0} \quad (7.24)$$

If we multiply on the left by \mathbf{u}_k^T , for $1 \leq k < j$, we get

$$\begin{aligned} 2\mathbf{u}_k^T \Sigma \mathbf{v} - 2\alpha \mathbf{u}_k^T \mathbf{v} - \beta_k \mathbf{u}_k^T \mathbf{u}_k - \sum_{\substack{i=1 \\ i \neq k}}^{j-1} \beta_i \mathbf{u}_k^T \mathbf{u}_i &= 0 \\ 2\mathbf{v}^T \Sigma \mathbf{u}_k - \beta_k &= 0 \\ \beta_k = 2\mathbf{v}^T \lambda_k \mathbf{u}_k = 2\lambda_k \mathbf{v}^T \mathbf{u}_k &= 0 \end{aligned}$$

where we used the fact that $\Sigma \mathbf{u}_k = \lambda_k \mathbf{u}_k$, as \mathbf{u}_k is the eigenvector corresponding to the k th largest eigenvalue λ_k of Σ . Thus, we find that $\beta_i = 0$ for all $i < j$ in Eq. (7.24), which implies that

$$\Sigma \mathbf{v} = \alpha \mathbf{v}$$

To maximize the variance along \mathbf{v} , we set $\alpha = \lambda_j$, the j th largest eigenvalue of Σ , with $\mathbf{v} = \mathbf{u}_j$ giving the j th principal component.

In summary, to find the best r -dimensional approximation to \mathbf{D} , we compute the eigenvalues of Σ . Because Σ is positive semidefinite, its eigenvalues must all be non-negative, and we can thus sort them in decreasing order as follows:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} \cdots \geq \lambda_d \geq 0$$

We then select the r largest eigenvalues, and their corresponding eigenvectors to form the best r -dimensional approximation.

Total Projected Variance

Let \mathbf{U}_r be the r -dimensional basis vector matrix

$$\mathbf{U}_r = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \\ | & | & & | \end{pmatrix}$$

with the projection matrix given as

$$\mathbf{P}_r = \mathbf{U}_r \mathbf{U}_r^T = \sum_{i=1}^r \mathbf{u}_i \mathbf{u}_i^T$$

Let \mathbf{A} denote the dataset formed by the coordinates of the projected points in the r -dimensional subspace, that is, $\mathbf{a}_i = \mathbf{U}_r^T \mathbf{x}_i$, and let $\mathbf{x}'_i = \mathbf{P}_r \mathbf{x}_i$ denote the projected point in the original d -dimensional space. Following the derivation for Eqs. (7.19), (7.21), and (7.22), the projected variance is given as

$$\text{var}(\mathbf{A}) = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i^T \mathbf{P}_r \mathbf{x}_i = \sum_{i=1}^r \mathbf{u}_i^T \Sigma \mathbf{u}_i = \sum_{i=1}^r \lambda_i$$

Thus, the total projected variance is simply the sum of the r largest eigenvalues of Σ .

Mean Squared Error

Based on the derivation for Eq. (7.23), the mean squared error objective in r dimensions can be written as

$$\begin{aligned}
 MSE &= \frac{1}{n} \sum_{i=1}^n \|\mathbf{x}_i - \mathbf{x}'_i\|^2 \\
 &= \text{var}(\mathbf{D}) - \text{var}(\mathbf{A}) \\
 &= \text{var}(\mathbf{D}) - \sum_{i=1}^r \mathbf{u}_i^T \boldsymbol{\Sigma} \mathbf{u}_i \\
 &= \text{var}(\mathbf{D}) - \sum_{i=1}^r \lambda_i
 \end{aligned}$$

The first r -principal components maximize the projected variance $\text{var}(\mathbf{A})$, and thus they also minimize the MSE.

Total Variance

Note that the total variance of \mathbf{D} is invariant to a change in basis vectors. Therefore, we have the following identity:

$$\text{var}(\mathbf{D}) = \sum_{i=1}^d \sigma_i^2 = \sum_{i=1}^d \lambda_i$$

Choosing the Dimensionality

Often we may not know how many dimensions, r , to use for a good approximation. One criteria for choosing r is to compute the fraction of the total variance captured by the first r principal components, computed as

$$f(r) = \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_r}{\lambda_1 + \lambda_2 + \cdots + \lambda_d} = \frac{\sum_{i=1}^r \lambda_i}{\sum_{i=1}^d \lambda_i} = \frac{\sum_{i=1}^r \lambda_i}{\text{var}(\mathbf{D})} \quad (7.25)$$

Given a certain desired variance threshold, say α , starting from the first principal component, we keep on adding additional components, and stop at the smallest value r , for which $f(r) \geq \alpha$. In other words, we select the fewest number of dimensions such that the subspace spanned by those r dimensions captures at least α fraction of the total variance. In practice, α is usually set to 0.9 or higher, so that the reduced dataset captures at least 90% of the total variance.

Algorithm 7.1 gives the pseudo-code for the principal component analysis algorithm. Given the input data $\mathbf{D} \in \mathbb{R}^{n \times d}$, it first centers it by subtracting the mean from each point. Next, it computes the eigenvectors and eigenvalues of the covariance matrix $\boldsymbol{\Sigma}$. Given the desired variance threshold α , it selects the smallest set of dimensions r that capture at least α fraction of the total variance. Finally, it computes the coordinates of each point in the new r -dimensional principal component subspace, to yield the new data matrix $\mathbf{A} \in \mathbb{R}^{n \times r}$.

ALGORITHM 7.1. Principal Component Analysis

PCA (\mathbf{D}, α):

- 1 $\boldsymbol{\mu} = \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i$ // compute mean
 - 2 $\mathbf{Z} = \mathbf{D} - \mathbf{1} \cdot \boldsymbol{\mu}^T$ // center the data
 - 3 $\boldsymbol{\Sigma} = \frac{1}{n} (\mathbf{Z}^T \mathbf{Z})$ // compute covariance matrix
 - 4 $(\lambda_1, \lambda_2, \dots, \lambda_d) = \text{eigenvalues}(\boldsymbol{\Sigma})$ // compute eigenvalues
 - 5 $\mathbf{U} = (\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_d) = \text{eigenvectors}(\boldsymbol{\Sigma})$ // compute eigenvectors
 - 6 $f(r) = \frac{\sum_{i=1}^r \lambda_i}{\sum_{i=1}^d \lambda_i}$, for all $r = 1, 2, \dots, d$ // fraction of total variance
 - 7 Choose smallest r so that $f(r) \geq \alpha$ // choose dimensionality
 - 8 $\mathbf{U}_r = (\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_r)$ // reduced basis
 - 9 $\mathbf{A} = \{\mathbf{a}_i \mid \mathbf{a}_i = \mathbf{U}_r^T \mathbf{x}_i, \text{ for } i = 1, \dots, n\}$ // reduced dimensionality data
-

Example 7.5. Given the 3-dimensional Iris dataset in Figure 7.1a, its covariance matrix is

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.681 & -0.039 & 1.265 \\ -0.039 & 0.187 & -0.320 \\ 1.265 & -0.32 & 3.092 \end{pmatrix}$$

The eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$ are given as

$$\begin{aligned} \lambda_1 &= 3.662 & \lambda_2 &= 0.239 & \lambda_3 &= 0.059 \\ \mathbf{u}_1 &= \begin{pmatrix} -0.390 \\ 0.089 \\ -0.916 \end{pmatrix} & \mathbf{u}_2 &= \begin{pmatrix} -0.639 \\ -0.742 \\ 0.200 \end{pmatrix} & \mathbf{u}_3 &= \begin{pmatrix} -0.663 \\ 0.664 \\ 0.346 \end{pmatrix} \end{aligned}$$

The total variance is therefore $\lambda_1 + \lambda_2 + \lambda_3 = 3.662 + 0.239 + 0.059 = 3.96$. The optimal 3-dimensional basis is shown in Figure 7.1b.

To find a lower dimensional approximation, let $\alpha = 0.95$. The fraction of total variance for different values of r is given as

r	1	2	3
$f(r)$	0.925	0.985	1.0

For example, for $r = 1$, the fraction of total variance is given as $f(1) = \frac{3.662}{3.96} = 0.925$. Thus, we need at least $r = 2$ dimensions to capture 95% of the total variance. This optimal 2-dimensional subspace is shown as the shaded plane in Figure 7.3a. The reduced dimensionality dataset \mathbf{A} is shown in Figure 7.4. It consists of the point coordinates $\mathbf{a}_i = \mathbf{U}_2^T \mathbf{x}_i$ in the new 2-dimensional principal components basis comprising \mathbf{u}_1 and \mathbf{u}_2 .

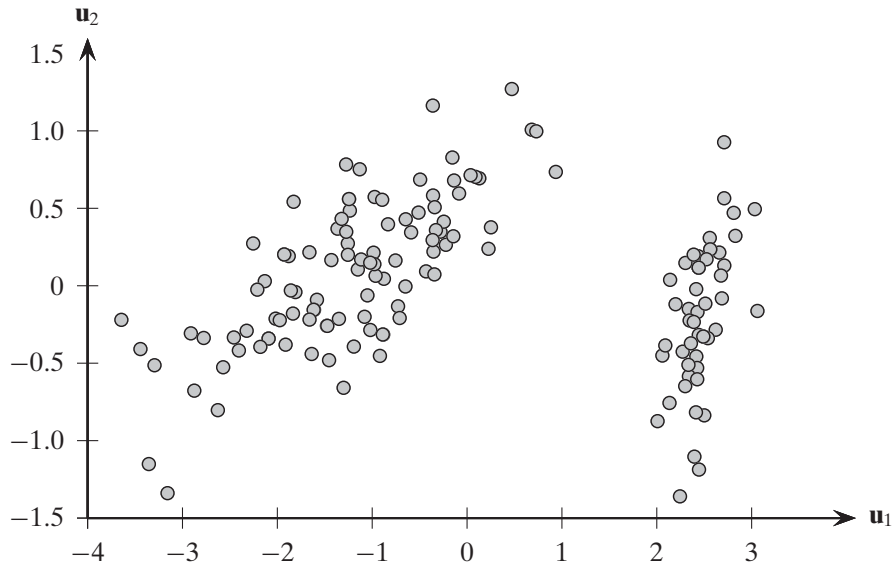


Figure 7.4. Reduced dimensionality dataset: Iris principal components.

7.2.4 Geometry of PCA

Geometrically, when $r = d$, PCA corresponds to a orthogonal change of basis, so that the total variance is captured by the sum of the variances along each of the principal directions $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_d$, and further, all covariances are zero. This can be seen by looking at the collective action of the full set of principal components, which can be arranged in the $d \times d$ orthogonal matrix

$$\mathbf{U} = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_d \\ | & | & \cdots & | \end{pmatrix}$$

with $\mathbf{U}^{-1} = \mathbf{U}^T$.

Each principal component \mathbf{u}_i corresponds to an eigenvector of the covariance matrix Σ , that is,

$$\Sigma \mathbf{u}_i = \lambda_i \mathbf{u}_i \text{ for all } 1 \leq i \leq d$$

which can be written compactly in matrix notation as follows:

$$\begin{aligned} \Sigma \begin{pmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_d \\ | & | & \cdots & | \end{pmatrix} &= \begin{pmatrix} | & | & \cdots & | \\ \lambda_1 \mathbf{u}_1 & \lambda_2 \mathbf{u}_2 & \cdots & \lambda_d \mathbf{u}_d \\ | & | & \cdots & | \end{pmatrix} \\ \Sigma \mathbf{U} &= \mathbf{U} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{pmatrix} \\ \Sigma \mathbf{U} &= \mathbf{U} \Lambda \end{aligned} \tag{7.26}$$