Swaption Valuation using Black76

by
Hammond Mason

1. Valuation of a Forward-Starting Interest Rate Swap

Before discussing how to value an *option* on an interest rate swap ("IRS”), it is worthwhile refreshing the reader's understanding of how an IRS is valued and, in particular, how to value an IRS that starts at some (certain) time in the future.

Simplistically, the value of an IRS is the difference between the present value (PV) of its two legs:

IRS Value = PV of Fixed Leg - PV of Floating Leg

1.1 Valuing the Fixed Leg

The value of the Fixed Leg is relatively easy. It is just the sum of the present values of a (fixed-rate) annuity. For a forward-starting swap, this value is:

$$\sum_{i=1}^{n}(Notional * ForwardStartingSwapRate * YearFraction_i * PVfactor_i)$$

where n is the number of payments on the Fixed Leg.

If the swap has a simple, flat, notional profile and you’re willing to dispense with day count and business day conventions and assume identical year fractions[1], then this formula simplifies to:

$$Notional * ForwardStartingSwapRate * YearFraction * \sum_{i=1}^{n} PVfactor_i$$

or

$$N * F * \frac{1}{m} * \sum_{i=1}^{n} PV_i$$

where

N is the notional principal of the swap
F is the forward-starting swap rate (“the Forward Swap Rate”) expressed as an annual rate with compounding matching m
m is the number of Fixed Leg payments per year, eg. “2” in the case of semi-annual amounts, “4" in the case of quarterly amounts, etc.
n is the number of payments in the annuity
PV_i is the PV factor for period i of the swap

1.2 Valuing the Floating Leg

At first glance, the value of the floating leg does not seem as simple. While the above formula applies equally to the floating leg as it does to the fixed leg, the problem arises in identifying what the floating interest rates are. Of course, these will only be known at the time they are determined and cannot be observed at the time of valuation. We must change our thinking to value the floating leg.

Consider a one-period floating rate bond. This bond has three cash flows:

1. purchase the bond;

[1] ie. accepting these assumptions prevents the ensuing explanation from becoming overly complicated.

2. receive the coupon; and
3. receive (redeem) the face value.

However, for an IRS, we are only concerned with the value of the coupon.

Since the purchase price of a bond represents the right to receive both the coupon and the redemption, if we remove the value of the redemption that is implicit in this purchase price, then what we have left is that part of the purchase price that pertains to the coupon.

So, if we buy the bond today and immediately sell the right to receive the redemption, our net cash flow is the price (today) of the coupon to be received in the future.

If someone came along and offered to pay us the same coupon in the future in return for a lower up-front amount (ie. price), then we could arbitrage this pricing difference by:

1. selling the bond;
2. immediately buying back the redemption;
3. using the net proceeds from the above to purchase the coupon at the abovementioned lower price;
4. keeping the difference; and
5. repeating 1-4 as much as possible until the vendor realises its mistake and stops offering the coupon at the lower price.

Similarly, if someone offered to buy the coupon from us at a higher price we could arbitrage by:

1. buying the bond;
2. immediately selling the redemption;
3. selling the coupon at the higher price to the buyer;
4. keeping the difference; and
5. repeating 1-4 as much as possible until the buyer realises its mistake.

Therefore, the price we have determined by buying the bond and selling its redemption, is the *arbitrage-free price* for the coupon.

The concept, of course, lends itself to the multi-period case, ie. an annuity of coupons, as illustrated below:

	t = 0	t = 1		t = T
Buy the bond	-£1			
Sell the redemption	+P(0,T)*£1			
Receive the coupon		+£c_1	...	+£c_n
Receive the redemption				+£1
Pay the redemption				-£1
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Net	-£1 + P(0,T)*£1	+£c_1	...	+£c_n

where P(0,T) is the present value at time t=0 of £1 received/paid at time t=T.

The important point to observe is that no matter how many coupons there are (ie. what the length of T is), the value of the floating coupon stream is always 1-P(0,T) multiplied by the face value of the bond (which, in our example is £1).

Now consider the case of a bond issued at some point in the future (t=T)[2]. The *no-arbitrage price* for the bond (coupons + redemption) is P(0,T). The value of the redemption is P(0,T+n). Therefore, using the same logic above, the value of the coupons-only annuity stream in P(0,T) – P(0,T+n).

The situation just described is identical to the case of a forward-starting IRS. Hence, the value of the Floating Leg of a forward-starting IRS is simply:

$$N * [P(0,T) - P(0,T+n)]$$

where

N	is the notional principal of the swap
$P(0,T)$	is the PV at t=0 of £1 received at t=T
$P(0,T+n)$	is the PV at t=0 of £1 received at t=T+n

1.3 Determining the Forward Swap Rate ("F")

If a forward-starting IRS is priced at par then its value is zero, ie. PV of Fixed Leg = PV of Floating Leg. The Forward Swap Rate, F, can then be determined using the above formulae:

PV of Fixed Leg = PV of Floating Leg

ie.

$$N * F * \frac{1}{m} * \sum_{i=1}^{n} PV_{T+i} = N * [P(0,T) - P(0,T+n)]$$

ie.[3]

$$F * \frac{1}{m} * \sum_{i=1}^{n} P(0,T+i) = [P(0,T) - P(0,T+n)]$$

ie.

$$F = \frac{P(0,T) - P(0,T+n)}{\frac{1}{m} * \sum_{i=1}^{n} P(0,T+i)}$$

2. Black76[4] and Swaptions

For illustration we will use the case of a payer swaption[5], ie. the holder of the swaption has the right to enter into an IRS and pay a fixed swap rate of K%pa. At the swaption's exercise date (T), if the market swap rate (S_T) is less than K%pa. then the holder will let the swaption lapse and, instead, enter into a swap with the market. If, however, S_T is greater than K%pa. then the holder will exercise the swaption.

Since the holder's position after exercise will be one of paying a fixed rate (K) lower than the market (S_T), the value of his/her position is equal to the sum of the present values of the differences between these payments, ie. the swaption payoff is:

[2] The first coupon occurs at t=T+1 and the last coupon occurs at t=T+n.
[3] The PV factor, PV_{T+i}, is the same as the price of a zero coupon bond that pays £1 at maturity, ie. $P(0,T+i)$.
[4] See F. Black, "The Pricing of Commodity Contracts", *Journal of Financial Economics*, Vol. 2, March 1976, pp. 167-179.
[5] The case for a receiver swaption is set out in Appendix A.

$$max\left[0, N * (S_T - K) * \frac{1}{m} * \sum_{i=1}^{n} P(T, T+i)\right]$$

re-arranging and recognising that $F_T = S_T$, gives:

$$max\left[0, \frac{N}{m} * \sum_{i=1}^{n} P(T, T+i) * (F_T - K)\right]$$

Applying Martingale Pricing Theory to this result[6] we arrive at the Black76 model for valuing a call (payer) swaption:

$$European\ Call\ (Payer\ Swaption) = C_t = [F.N(d_1) - K.N(d_2)].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

where

$$d_1 = \frac{\ln\frac{F}{K} + \frac{\sigma^2}{2}.(T-t)}{\sigma.\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma.\sqrt{T-t}$$

and

F	is the Forward Swap Rate (observed at time t=t)
$N()$	is the cumulative standard Normal distribution
K	is the strike rate
N	is the notional principal of the swap
n	is the number of payments under the fixed leg of the swap
m	is the number of fixed leg payments per year (as before, we are ignoring day count and business day conventions and assuming all periods are of equal length)
$P(t, T+i)$	is the PV at time t=t of £1 received at time t=T+i
σ	is the volatility of F
T	is the option exercise date (=swap start date)
t	is the date we are valuing the option (t=0 at option inception)

3. Swaption Delta

$$Delta = \frac{\partial C}{\partial F}$$

$$= \frac{\partial}{\partial F}\left[[F.N(d_1) - K.N(d_2)].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)\right]$$

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\frac{\partial}{\partial F}[F.N(d_1) - K.N(d_2)]$$

[6] which recognises that the value of an option is equal to the expectation (taken with respect to, in this case, the Forward Neutral Measure) of the present values of all possible payoffs.

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\left\{\frac{\partial}{\partial F}[\mathrm{F}.N(d_1)] - \frac{\partial}{\partial F}[K.N(d_2)]\right\}$$

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\left\{\left[\mathrm{F}.\frac{\partial N(d_1)}{\partial F} + N(d_1)\right] - \left[K.\frac{\partial N(d_2)}{\partial F}\right]\right\}$$

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\left\{N(d_1) + \mathrm{F}.\frac{\partial N(d_1)}{\partial F} - K.\frac{\partial N(d_2)}{\partial F}\right\}$$

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\left\{N(d_1) + F.\left[\frac{\partial N(d_1)}{\partial d_1}.\frac{\partial d_1}{\partial F}\right] - K.\left[\frac{\partial N(d_2)}{\partial d_2}.\frac{\partial d_2}{\partial F}\right]\right\}$$

since $\frac{\partial N(d_2)}{\partial d_2} = \frac{\partial N(d_1)}{\partial d_1}.\frac{F}{K}$ and $\frac{\partial \mathrm{d}_2}{\partial \mathrm{F}} = \frac{\partial \mathrm{d}_1}{\partial \mathrm{F}}$ we get[7]:

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\left\{N(d_1) + F.\frac{\partial N(d_1)}{\partial d_1}.\frac{\partial d_1}{\partial F} - K.\left[\frac{\partial N(d_1)}{\partial d_1}.\frac{F}{K}\right].\frac{\partial d_1}{\partial F}\right\}$$

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).\left\{N(d_1) + \left[F.\frac{\partial N(d_1)}{\partial d_1}.\frac{\partial d_1}{\partial F} - F.\frac{\partial N(d_1)}{\partial d_1}.\frac{\partial d_1}{\partial F}\right]\right\}$$

$$= \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i).N(d_1)$$

$$= N(d_1).\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

[7] See Appendix B for the proof.

Appendix A: Receiver Swaption

The holder of a receiver swaption has the right to enter into an IRS and receive a fixed swap rate of K%pa. At the swaption's exercise date (T), if the market swap rate (S_T) is greater than K%pa. then the holder will let the swaption lapse and, instead, enter into a swap with the market. If, however, S_T is less than K%pa. then the holder will exercise the swaption.

Since the holder's position after exercise will be one of receiving a fixed rate (K) higher than the market (S_T), the value of his/her position is equal to the sum of the present values of the differences between these payments, ie. the swaption payoff is:

$$max\left[0, N*(K-S_T)*\frac{1}{m}*\sum_{i=1}^{n}P(T,T+i)\right]$$

re-arranging and recognising that $F_T = S_T$, gives:

$$max\left[0, \frac{N}{m}*\sum_{i=1}^{n}P(T,T+i)*(K-F_T)\right]$$

Applying Martingale Pricing Theory to this result we arrive at the Black76 model for valuing a put (receiver) swaption:

$$European\ Put\ (Receiver\ Swaption) = P_t = [-F.N(-d_1)+K.N(-d_2)].\frac{N}{m}.\sum_{i=1}^{n}P(t,T+i)$$

where

$$d_1 = \frac{\ln\frac{F}{K}+\frac{\sigma^2}{2}.(T-t)}{\sigma.\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma.\sqrt{T-t}$$

and

F	is the Forward Swap Rate (observed at time t=t)
$N()$	is the cumulative standard Normal distribution
K	is the strike rate
N	is the notional principal of the swap
n	is the number of payments under the fixed leg of the swap
m	is the number of fixed leg payments per year (as before, we are ignoring day count and business day conventions and assuming all periods are of equal length)
$P(t,T+i)$	is the PV at time t=t of £1 received at time t=T+i
σ	is the volatility of F
T	is the option exercise date (=swap start date)
t	is the date we are valuing the option (t=0 at option inception)

Given the symmetry of the Normal distribution, $N(-d) = 1 - N(d)$, so the above equation can be restated as:

$$P_t = [-F.\{1-N(d_1)\}+K.\{1-N(d_2)\}].\frac{N}{m}.\sum_{i=1}^{n}P(t,T+i)$$

$$= [-F + F.N(d_1) + K - K.N(d_2)].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

$$= [K - F + F.N(d_1) - K.N(d_2)].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

$$= [K - F].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i) + [F.N(d_1) - K.N(d_2)].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

$$= [K - F].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i) + C_t$$

$$= C_t + \ [K - F].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

which leads to the Put-Call Parity result:

$$P_t - C_t = [K - F].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

and also gives the delta of a receiver swaption:

$$Delta = \frac{\partial P}{\partial F}$$

$$= \frac{\partial C}{\partial F} + \frac{\partial}{\partial F}\left[(K - F).\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)\right]$$

$$= N(d_1).\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i) - \frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

$$= [N(d_1) - 1].\frac{N}{m}.\sum_{i=1}^{n} P(t, T+i)$$

Appendix B: Proof that $\frac{\partial N(d_2)}{\partial d_2} = \frac{\partial N(d_1)}{\partial d_1}.\frac{F}{K}$ and $\frac{\partial d_2}{\partial F} = \frac{\partial d_1}{\partial F}$

$$\frac{\partial N(d_2)}{\partial d_2} = \frac{\partial}{\partial d_2}\left[\frac{1}{\sqrt{2.\pi}}.\int_{-\infty}^{d_2} e^{-\frac{1}{2}z^2}.dz\right]$$

$$= \frac{e^{-\frac{1}{2}.(d_2)^2}}{\sqrt{2.\pi}}$$

$$= \frac{e^{-\frac{1}{2}.\left(d_1-\sigma.\sqrt{T-t}\right)^2}}{\sqrt{2.\pi}}$$

$$= \frac{e^{-\frac{1}{2}.\left[(d_1)^2-2.d_1.\sigma.\sqrt{T-t}+\sigma^2.(T-t)\right]}}{\sqrt{2.\pi}}$$

$$= \frac{e^{-\frac{1}{2}.(d_1)^2}.e^{d_1.\sigma.\sqrt{T-t}}.e^{-\frac{\sigma^2}{2}.(T-t)}}{\sqrt{2.\pi}}$$

$$= \frac{e^{-\frac{1}{2}.(d_1)^2}.e^{\ln\left(\frac{F}{K}\right)+\frac{\sigma^2}{2}.(T-t)}.e^{-\frac{\sigma^2}{2}.(T-t)}}{\sqrt{2.\pi}}$$

$$= \frac{e^{-\frac{1}{2}.(d_1)^2}.e^{\ln\frac{F}{K}}}{\sqrt{2.\pi}}$$

$$= \frac{e^{-\frac{1}{2}.(d_1)^2}}{\sqrt{2.\pi}}.\frac{F}{K}$$

$$= \frac{\partial N(d_1)}{\partial d_1}.\frac{F}{K}$$

$$\frac{\partial d_2}{\partial F} = \frac{\partial}{\partial F}\left[d_1 - \sigma.\sqrt{T-t}\right]$$

$$= \frac{\partial d_1}{\partial F} - \frac{\partial}{\partial F}\left[\sigma.\sqrt{T-t}\right]$$

$$= \frac{\partial d_1}{\partial F} - 0$$

$$= \frac{\partial d_1}{\partial F}$$